169

ELIMINATION

BETWEEN

TWO UNKNOWN EQUATIONS WITH TWO UNKNOWN QUANTITIES,

BY MEANS OF

THE GREATEST COMMON DIVISOR.

ALSO,

ANALYSIS OF CURVES,

WITH AN APPLICATION TO

AN EQUATION OF THE FOURTH DEGREE.

BY

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PREFACE.

The following pages consist of two demonstrations, which are intended to explain and illustrate parts of the Course of Mathematics which are only imperfectly discussed in Elementary Text-Books:

1. *Elimination between two equations with two unknown quantities, by means of the greatest common divisor.*
2. *Analysis of Curves—with an application to the discussion of an equation of the fourth degree.*

The first article is taken principally from Garnier and De Fourcy; and the second is a translation of the admirable discussion of Lacroix, in his *Calcul Différentiel et Intégral.*

Virginia Military Institute,
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RESOLUTION, &c.

*Resolution of two equations with two unknown quantities.—
Elimination by the Greatest Common Divisor.*

1. The most general equation of the mth degree, between two unknown quantities x and y, contains all the terms in which the sum of the exponents of x and y does not exceed m. Its form may then be represented by the equation

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} \ldots . + Tx + u = o,$$

in which P, Q, R, &c. are functions of y, as follows:

P represents a polynomial of the first degree in y of the form $a + by$

Q represents a polynomial of the second degree in y of the form $c + dy + ey^2$

R represents a polynomial of the third degree in y of the form $f + gy + hy^2 + ly^3$.

&c. &c. &c.

the last co-efficient u, containing all the powers of y, from zero to m.

2

2. An equation thus formed is said to be a *complete* equation of the mth degree between two unknown quantities, and when any of its terms are wanting, it is called an *incomplete* equation.

3. Could we solve equations of every degree, the ordinary methods of elimination might be readily applied, to the solution of any system of m equations, with m unknown quantities; and we should, in general, obtain a determinate number of solutions. It would be only necessary to find the value of one of the unknown quantities in terms of the others, in one of the equations, and substitute this value in each of the other equations; there would result a new system of equations, with one less equation than were given, and with one unknown quantity less. By continuing this operation, we should obtain a single equation with but one unknown quantity. This equation is called the *final* equation, and serves to determine the values of the unknown quantity which it contains, which by substitution will make known the corresponding values of the others.

4. If the number of equations exceeded the number of unknown quantities, we could by the above method eliminate all the unknown quantities, and there would result one or more equations, containing only known terms, which would be *equations of condition* necessary to be fulfilled, in order that the given equations should not be incompatible with each other.

5. Should the number of unknown quantities exceed the number of equations, the question would be *indeterminate;* for by giving arbitrary values to as many of the unknown quantities as were in excess, we might determine the values

of the others by means of the given equations, and thus have as many different solutions as there were arbitrary values assumed.

6. But the difficulty of solving equations in general, has led algebraists to seek other methods of elimination than the one just mentioned, so as to obtain at once a single equation involving but one unknown quantity. Various methods have been used to determine this final equation, and that method is regarded as best, which leads to a final equation, whose roots make known all the values of the unknown quantity which it contains, which are compatible with the given equations, and no other values. The method by the *Greatest Common Divisor* is not free from the objection of leading to foreign roots, but it is found to be the most convenient in practice. We propose to explain this method.

7. Let

$$A = o \qquad\qquad B = o$$

be two equations involving x and y, and let β be any assumed value of y. If we substitute this value in the place of y, in the given equations, there will result two equations,

$$A' = o \qquad\qquad B' = o$$

which contain only x and known quantities.

Now it is evident that β can only satisfy the given equations, when there exists at least one value of x, which will reduce the two quantities A' and B' to zero at the same time, that is, satisfy the equations

$$A' = o \qquad B' = o.$$

Let $x = \alpha$ be such a value of x, it follows, that these last equations must have a common divisor in x, since they will both be divisible by $(x - \alpha)$, if α be a common root. When this condition is fulfilled, the value of y is called a *compatible* value. Hence,

Having given two equations with two unknown quantities, a value attributed to one of the unknown quantities will be compatible, when its substitution in the given equations causes them to have a common divisor, which is a function of the other unknown quantity.

8. The above principle leads directly to the method to be pursued, to effect the resolution of the given equations. For, since every compatible value of one of the unknown quantities, y, for example, must when substituted give a common divisor in x, (if the two equations be determinate,) it follows, that if the proposed equations be arranged with reference to x, and we seek the greatest common divisor to the polynomials composing them, we shall, after successive divisions, find, *in general*, a remainder which contains only y and known quantities. This remainder must be zero, if the given equations have a common divisor in x. Calling R this remainder, we shall by placing it equal to zero, form the equation

$$R = o,$$

which is the *final* equation spoken of. This equation expresses the condition necessary for the common divisor in x to exist. The roots of this equation substituted in the given

equations, will cause them to present the same values for x, and of course to have a common divisor in x, if no *foreign* roots have been introduced in the process of finding the greatest common divisor. The method of detecting and removing these foreign roots will presently be examined.

9. It will generally be sufficient to substitute the values of y deduced from the final equation, in the last divisor in x, in order to obtain the corresponding values of x. For, if we represent by Q the quotient resulting from the division of A by B, and by R, the first remainder, we shall have

$$A = B \times Q + R.$$

From this equation it is evident, that if any values of x and y reduce A and B to zero, they must make $R = o$ also. The equations

$$B = o \qquad\qquad R = o$$

will therefore make known the values of x and y, which will satisfy the three equations

$$A = o \qquad\qquad B = o \qquad\qquad R = o.$$

Reciprocally, the values of x and y, which satisfy the two last equations, will also satisfy the equation

$$A = o.$$

The determination of the roots of the given equation is then reduced to that of the equations

$$B = o \qquad R = o.$$

Dividing now B by R, we shall obtain an equation

$$B = R \times Q' + R'.$$

The roots of the equations

$$B = o \qquad R = o$$

will therefore be found among the solutions of the equations

$$R = o \qquad R' = o.$$

And if R′ be the remainder in y,

$$R' = o$$

will be the final equation, the roots of which substituted in

$$R = o$$

will make known the systems of values which correspond to the equations

$$R' = o \qquad R = o \qquad B = o \qquad A = o$$

and of course the values which are compatible in the given equations.

10. Let us apply the foregoing principles to the following example:

EXAMPLE I.

$$A = x^3 - 3yx^2 + (3y^2 - y + 1)\,x - y^3 + y^2 - 2y = o$$

$$B = x^2 - 2yx + y^2 - y = o.$$

Following the rule for obtaining the greatest common divisor, we have,

First Division.

$$\begin{array}{l|l}
x^3 - 3yx^2 + (3y^2 - y + 1)\,x - y^3 + y^2 - 2y & x^2 - 2yx + y^2 - y \\
\underline{x^3 - 2yx^2 + (y^2 - y)\,x} & x - y = Q \\
\quad - yx^2 + (2y^2 + 1)\,x - y^3 + y^2 - 2y & \\
\quad \underline{- yx^2 + 2y^2x - y^3 + y^2} & \\
\qquad\qquad\qquad x - 2y = R. &
\end{array}$$

Second Division.

$$\begin{array}{l|l}
x^2 - 2yx + y^2 - y & x - 2y \\
\underline{x^2 - 2yx} & x = Q' \\
\qquad\quad \underline{y^2 - y} = R' &
\end{array}$$

In order that $x - 2y$ be a common divisor to the two

given equations, the last remainder must be zero. We have therefore for the final equation,

$$y^2 - y = o.$$

The roots of this equation are

$$y = o \qquad y = 1.$$

Substituting them in the equation

$$x - 2y = o,$$

formed by placing the last divisor equal to zero (9), we find the corresponding values of x to be

$$x = o \qquad x = 2,$$

which determine the solutions of the given equations.

11. Should the quotient resulting from the division of A by B, be a fraction, the denominator of which contained either or both of the unknown quantities, the principle developed in Art. 9. would no longer hold good. For if in the equation

$$A = B \times Q + R,$$

Q were equal to $\frac{H}{K}$, K containing one or both of the un-

known quantities, we should have

$$A = \frac{BH}{K} + R.$$

The values of x and y, which reduce A and B to zero, might also cause K to be zero. $\frac{BH}{K}$ would then become $\frac{o}{o}$, and might have a finite or infinite value. The value of R would also be finite or infinite, and could not in either case be zero. The roots of the given equations could not then be found from the solution of the equations

$$B = o \qquad\qquad R = o.$$

12. To avoid fractional quotients, we adopt the same expedients resorted to, in obtaining the greatest common divisor, and which consist, either in suppressing the common factors of the dividend or divisor, or by multiplying by some factor which will render the division possible. We shall thus be enabled always to obtain entire quotients, in which case the method of Art. 9 may be followed.

13. In seeking the greatest common divisor to two polynomials, the suppression of common factors, or the introduction by multiplication of a new factor, does not affect the result, but it is not the case in the solution of equations. We shall therefore examine the consequences resulting from the introduction or suppression of these factors.

14. Let us take the equations

$$A = o \qquad\qquad B = o,$$

and let us suppose that the division of A by B cannot be effected; which supposes that the co-efficient of the first term of B contains factors of y which are not common to that of the first term of A. Let D be the product of all these factors, and suppose D common to all the terms of B. The proposed equations will take the form

$$A = o \qquad B'D = o.$$

These equations may be satisfied by making

$$A = o \qquad B' = o.$$

or

$$A = o \qquad D = o,$$

We might then suppress the common factor D, provided that to the solutions of the equations

$$A = o \qquad B' = o,$$

we add those of

$$A = o \qquad D = o.$$

To obtain the solutions of the last equations, we find the values of y in the equation

$$D = o,$$

and substitute them in

$$A = o;$$

the systems of values of x and y, thus obtained, will give all the solutions belonging to the given equations.

15. Let us now suppose that D is not common to all the terms of the divisor, and that an entire quotient can only be obtained by multiplying the dividend by D. We shall then have

$$AD = o \qquad B = o.$$

These equations may be satisfied by either of the systems of equations

$$A = o \qquad B = o, \qquad \text{or} \qquad D = o \qquad B = o.$$

Hence the system of values of x and y deduced from the equations

$$D = o \qquad B = o,$$

must be suppressed as not belonging to the solutions of the given equations, if they are found existing among the solutions of the final equations.

16. The following example will illustrate the case alluded to in Art. 14:

EXAMPLE II.

$$yx^5 + 3yx^3 - y^2x^2 + (y + 1)\,x - y = o \;.\;.\; (1)$$

$$yx^4 + (3y - 1)\,x^3 - y^2x + y - 1 = o \;.\;.\;.\; (2).$$

The result of the first division gives for a remainder

$$x^3 + 2x - y.$$

Dividing equation (2) by this remainder, we have for the second remainder

$$(y - 1)\,x^2 + (y - 1).$$

This remainder having the common factor $(y - 1)$, we suppress this factor (Art. 14), and proceeding in the operation, we have for the last divisor $(x - y)$, and for the final equation

$$y^2 + 1 = o.$$

The roots of this equation are

$$y = +\sqrt{-1} \qquad y = -\sqrt{-1}.$$

Substituting these values in the equation formed by putting the last divisor equal to zero,

$$x - y = o,$$

we have for the corresponding values of x

$$x = +\sqrt{-1} \qquad x = -\sqrt{-1}.$$

Making now the suppressed factor $(y - 1)$ equal to zero (Art. 14), the equation

$$y - 1 = o,$$

gives $y = 1$, which being substituted in the preceding divisor placed equal to zero, viz:

$$x^2 + 2x - y = o.$$

The values of x deduced from this equation, will give the solutions to the given equations which were omitted in the suppression of the common factor $(y - 1)$.

17. For an application of Art. 15, take the example:

EXAMPLE III.

$$(y - 1)x^2 + 2x - 5y + 3 = o \quad . \; . \; . \; . \quad (1)$$

$$yx^2 + 9x - 10y = o \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

To render the division possible, we multiply the polynomial in equation (1) by y, and going through the operation of division we have a remainder

$$(-7y+9)x+5y^2-7y.$$

Since we have introduced a factor in the dividend, we must examine whether or not any foreign roots have been added to the question. Taking the equations (Art. 15),

$$y=o \qquad yx^2+9x-10y=o,$$

we find their roots to be

$$y=o \qquad\qquad x=o.$$

Should the final equations give these values among their solutions, they must be rejected, as not belonging to the given equations.

Proceeding now to the second division, after multiplying the last divisor by $-7y+9$, to render the division possible, we obtain for the last divisor

$$(-7y+9)x+5y^2-7y,$$

and for the final equations

$$25y^5-70y^4-126y^3+414y^2-243y=o.$$

The roots of which are found by known rules to be

$$y = o,\ y = 1,\ y = 3,\ y = \frac{-3 + 3\sqrt{10}}{5},\ y = \frac{-3 - 3\sqrt{10}}{5}.$$

The corresponding values of x deduced from the equation

$$(-7y + 9)\,x + 5y^2 - 7y = o,$$

being

$$x = o,\ x = 1,\ x = 2,\ x = -5 - \sqrt{10},\ x = -5 + \sqrt{10}.$$

Here we see we have the solutions $x = o$, $y = o$, found above. They must then be rejected, and as the multiplication in the second division by $-7y + 9$ has not introduced any foreign roots, the proposed equations admit of the four following solutions :

$$1\left\{\begin{array}{l} y = 1 \\ x = 1 \end{array}\right. \qquad 3\left\{\begin{array}{l} y = \dfrac{-3 + 3\sqrt{10}}{5} \\ x = -5 - \sqrt{10} \end{array}\right.$$

$$2\left\{\begin{array}{l} y = 3 \\ x = 2 \end{array}\right. \qquad 4\left\{\begin{array}{l} y = \dfrac{-3 - 3\sqrt{10}}{5} \\ x = -5 + \sqrt{10}. \end{array}\right.$$

If we had substituted the roots $x = o$, $y = o$, in equation (1), in the first place, we should have seen they would not satisfy it, and we might have concluded that they formed no part of the solution of the question.

18. Examples are frequently presented in which it is necessary in the process for finding the greatest common divisor, to introduce as well as to suppress factors, to render the division possible. The foregoing directions must be observed, and all compatible values which are thus suppressed must be joined to the solutions given by the final equations; while those which have been introduced must be rejected.

19. If the given equations can be decomposed into common factors, their resolution will be very much simplified by putting these common factors equal to zero, separately. There may be two cases, 1st. the common factor may be a function of one of the unknown quantities only; 2nd. It may contain both.

20. Let us examine the first case. Take the two equations

$$A = o \qquad\qquad B = o,$$

and suppose them to contain a common factor which is a function of x, we may substitute for the equations the following:

$$f(x) \times F(x, y) = o \;.\;.\;.\;.\; (1)$$

$$f(x) \times \varphi(x, y) = o \;.\;.\;.\;.\; (2).$$

These equations may be satisfied by making

$$f(x) = o.$$

As this equation contains only x and known terms, it

will give a determinate number of values of x, which will satisfy the given equations independently of any determination of y.

But the given equations may be satisfied by either of the following hypotheses, viz:

$$f(x) = o \text{ and } \varphi(x, y) = o \quad . \; . \; . \; . \quad (3)$$

$$f(x) = o \text{ and } F(x, y) = o \quad . \; . \; . \; . \quad (4)$$

or

$$F(x, y) = o \quad \text{and } \varphi(x, y) = o \quad . \; . \; . \; . \quad (5).$$

But the solutions resulting from the systems (3) and (4) do not differ from those determined by the equation

$$f(x) = o,$$

for the values of x resulting from this equation, will when substituted in the equations

$$\varphi(x, y) = o \qquad F(x, y) = o,$$

lead to a determinate number of values of y, which will be included in the solutions of equation

$$f(x) = o,$$

the roots of which will satisfy the given equations for any values of y. Hence, to determine the remaining solutions of the question, we have to obtain the values of x and y resulting from the condition (5).

$$\mathrm{F}\,(x, y) = o \qquad\qquad \varphi\,(x, y) = o,$$

the values of which we can determine by the ordinary rule.

21. If now the common factor contain both x and y, the two equations will assume the form

$$f\,(x, y) \times \mathrm{F}\,(x, y) = o$$

$$f\,(x, y) \times \varphi\,(x, y) = o.$$

Making in the first place

$$f\,(x, y) = o,$$

the two equations will be satisfied. This equation shows (Art. 5,) that by assuming any values of y, we shall have a determinate number of values of x, and reciprocally, the values of x being assumed, those of y will be determined. The equations therefore admit of an indefinite number of solutions, resulting from the presence of the common factor, $f\,(x, y)$.

But the hypotheses of

$$f\,(x, y) = o \quad \text{and } \varphi\,(x, y) = o$$

and

$$f\,(x, y) = o \quad \text{and } \mathrm{F}\,(x, y) = o,$$

will also satisfy the given equations. These equations cannot however give new solutions to the question, since they

will necessarily be included in those which result from the condition

$$f(x, y) = o.$$

To determine new solutions, we must therefore take the final conditions

$$\varphi(x, y) = o \qquad F(x, y) = o,$$

and determine the values of x and y, as in ordinary cases.

22. To apply the above principles, take the following example:

EXAMPLE IV.

$$(x^2 + y^2)(yx - 6)(x - 1) = o$$

$$(x^2 + y^2)(2x - 3y)(x - y) = o.$$

These equations give an indefinite number of solutions (Art. 21) in consequence of the common factor $(x^2 + y^2)$. Suppressing this factor, we have the two equations

$$(yx - 6)(x - 1) = o$$

$$(2x - 3y)(x - y) = o.$$

These equations are satisfied by either of the following systems of equations:

$$yx - 6 = o \qquad 2x - 3y = o$$

$$yx - 6 = o \qquad x - y = o$$

$$x - 1 = o \qquad 2x - 3y = o$$

$$x - 1 = o \qquad x - y = o.$$

From which we deduce the following additional solutions:

$$y = +2 \qquad x = +3$$

$$y = -2 \qquad x = -3$$

$$y = \pm\sqrt{6} \qquad x = \pm\sqrt{6}$$

$$y = +\tfrac{2}{3} \qquad x = +1$$

$$y = +1 \qquad x = +1.$$

23. EXAMPLE V.

Let us take the equations

$$y\,(x - 1)\,(x + y) \times (x + 1)\,(x^2 - 2y - 1) = o$$

$$y\,(x - 1)\,(x + y) \times y\,(x^2 - y^2) = o.$$

These equations have three common factors, viz: y, $x - 1$, $x + y$.

Placing them separately equal to zero, we have the three equations,

$$y = o \qquad x - 1 = o \qquad x + y = o.$$

The first result shows (Art. 20) that the given equations will be satisfied by a value of $y = o$, independently of any determination of x; the second gives a root of $x = 1$, y being indeterminate; while the third shows (Art. 21) that for the common factor $(x + y)$, the given equations admit of an indefinite number of solutions.

We have therefore the following solutions from these common factors,

$$\begin{cases} y = o \\ x \text{ indeterminate} \end{cases} \quad \begin{cases} x = 1 \\ y \text{ indeterminate} \end{cases} \quad \begin{cases} x = -y \\ y \text{ indeterminate.} \end{cases}$$

The given equations present in addition the following systems of equations:

$$\begin{array}{ll} y = o & x + 1 = o \\ y = o & x^2 - 2y - 1 = o \\ x^2 - y^2 = o & x + 1 = o \\ x^2 - y^2 = o & x^2 - 2y - 1 = o, \end{array}$$

which give the solutions

$$\begin{array}{ll} 1.\ y = o & x = -1 \\ 2.\ y = o & x = +1 \\ 3.\ y = o & x = -1 \\ 4.\ y = +1 & x = -1 \end{array}$$

5. $y = -1 \qquad x = -1.$

6. $y = 1 + \sqrt{2} \quad x = 1 + \sqrt{2}$

7. $y = 1 + \sqrt{2} \quad x = -1 - \sqrt{2}$

8. $y = 1 - \sqrt{2} \quad x = 1 - \sqrt{2}$

9. $y = 1 - \sqrt{2} \quad x = -1 + \sqrt{2}.$

Solutions (1), (2), and (3) are included in the solution

$$y = o \quad x \text{ indeterminate,}$$

resulting from the common factor, y, while solutions (4), (7), and (9) are given by the equation

$$x = -y.$$

We have therefore the following new solutions only,

$$\begin{cases} y = -1 \\ x = -1 \end{cases} \quad \begin{cases} y = 1 + \sqrt{2} \\ x = 1 + \sqrt{2} \end{cases} \quad \begin{cases} y = 1 - \sqrt{2} \\ x = 1 - \sqrt{2}. \end{cases}$$

24. EXAMPLE VI.

$$x^2 + (8y - 13)\,x + y^2 - 7y + 12 = o$$

$$x^2 - (4y + 1)\,x + y^2 + 5y = o.$$

First Division.

$$
\begin{array}{l|l}
x^2 + (8y - 13)\,x + y^2 - 7y + 12 & x^2 - (4y + 1)\,x + y^2 + 5y \\
\cline{2-2}
x^2 - (4y + 1)\,x + y^2 + 5y & \quad 1 \\
\cline{1-1}
\quad (12y - 12)\,x - 12y + 12. &
\end{array}
$$

This remainder can be decomposed into factors, as follows :

$$12\,(y - 1)\,(x - 1).$$

The question is thus reduced to the solution of the following system of equations :

$$1.\ \begin{cases} y - 1 = o \\ x^2 - (4y + 1)\,x + y^2 + 5y = o \end{cases}$$

$$2.\ \begin{cases} x - 1 = o \\ x^2 - (4y + 1)\,x + y^2 + 5y = o. \end{cases}$$

The solutions of which are readily found to be

$$
\begin{array}{ll}
y = 1 & \quad x = 3 \\
y = 1 & \quad x = 2 \\
y = o & \quad x = 1 \\
y = -1 & \quad x = 1
\end{array}
$$

25. When the final equation is independent of y, and contains known terms only, which do not of themselves reduce to zero, the given equations are *contradictory*, and cannot be

satisfied by the same values of x and y; for, the condition of their having common values, requires that they should have a common divisor (Art. 7), and this common divisor cannot exist where the final equation is not satisfied.

The following example will illustrate this principle.

EXAMPLE VII.

$$yx^3 - (y^3 - 3y - 1)\,x + y = o$$

$$x^2 - y^2 + 3 = o.$$

First Division.

$$\begin{array}{r|l} yx^3 - (y^3 - 3y - 1)\,x + y & \underline{x^2 - y^2 + 3} \\ \underline{yx^3 - (y^3 - 3y)\,x \qquad\quad} & yx \\ \underline{x + y.} & \end{array}$$

Second Division.

$$\begin{array}{r|l} x^2 - y^2 + 3 & \underline{x + y} \\ \underline{x + xy \qquad\quad} & x - y \\ - xy - y^2 + 3 & \\ \underline{- xy - y^2 \qquad} & \\ + 3. & \end{array}$$

The last remainder being 3, the final equation is

$$3 = o.$$

But this equation is absurd, since 3 cannot be equal to zero. The proposed equations have therefore no common divisor, and are consequently contradictory.

26. Should the final equation reduce to zero, of itself, the given equations will contain a common divisor independently of any determination of y. If this common divisor contain only one of the unknown quantities, x for example, the equations would be satisfied by a definite number of values of x, y being indeterminate (Art. 20); while they would admit of an infinite number of solutions if it contained both x and y (Art. 21). Take the following equations:

EXAMPLE VIII.

$$x^3 - 3yx^2 + 3y^2x - 5x^2 + 10yx + 6x - y^3 - 5y^2 - 6y = o$$

$$x^3 - 5yx^2 + 8y^2x - x - 4y^3 + y = o.$$

After the third division we obtain for the common divisor, there being no remainder,

$$(y^4 - 10y^3 + 35y^2 - 50y + 24)x - y^5 + 10y^4 - 35y^3 + 50y^2 - 24y.$$

The proposed equations have therefore a common factor, and by putting the above common divisor equal to zero, we have

$$x = y.$$

Hence $x - y$ is a common factor to the two equations, and they therefore admit of an indefinite number of solutions (Art. 21.)

If we divide the given equations by this common factor, we shall have the two equations,

$$x^2 - (2y + 5)\,x + y^2 + 5y + 6 = o$$

$$x^2 - 4yx + 4y^2 - 1 = o.$$

Operating upon these equations by the ordinary rule, we shall have for the last divisor in x,

$$(2y - 5)\,x + 5y - 3y^2 + 7,$$

and for the final equation,

$$y^4 - 10y^3 + 35y^2 - 50y + 24 = o\,;$$

from which we deduce the following solutions,

$$\begin{array}{ll} y = 1 & x = 3 \\ y = 2 & x = 5 \\ y = 3 & x = 5 \\ y = 4 & x = 7. \end{array}$$

27. The substitution of the values of y deduced from the final equation in the divisor of the first degree in x, may cause the values of x to assume either of the following forms, viz:

$$(1)\ x = a,\ (2)\ x = o,\ (3)\ x = \theta,\ (4)\ x = \frac{o}{o}.$$

28. Represent the divisor in x by $\mathrm{A}x - \mathrm{B}$, A and B be-

ing functions of y. In the first case, if $y = B$ be the value of y, which by substitution in the equation

$$Ax - B = o,$$

gives $x = \alpha$, the given equations admit of but this value x, corresponding to the value of $y = \beta$; since the equation from which the value of x is obtained is of the *first* degree only, and can give but one solution. This is also the case when the value of y gives $x = o$.

29. In the third case, when we find $x = \theta$, for the value of $y = \beta$, the two equations are contradictory; for the equation

$$Ax - B = o$$

can only give $x = \theta$, when the substitution of the value of y makes $A = o$ and B equal to a finite quantity.

But when $A = o$, we have from the nature of the above equation, $B = o$ also. Hence the equation

$$Ax - B = o$$

is absurd, for $x = \theta$ and $y = \beta$. Further, the number B is the common divisor which the substitution of $y = \beta$ causes the given equations to acquire, and if all the values of y produce in the same manner a numerical common divisor, it is evident no values of x and y can satisfy the conditions of the questions. The proposed equations are therefore contradictory.

30. Finally, if $y = \beta$ reduce A and B to zero, at the same time, the value of x becomes $\frac{o}{o}$ or *indeterminate.*

This result shows that the equation formed by placing the divisor of the first degree equal to zero, does not make known all the values of x, which will satisfy the proposed equations for the value of $y = \beta$, since this equation reduces to zero, by the substitution of this value of y, independently of any determination of x. It is therefore *indeterminate,* and if the value of y be substituted in the given equations, it will cause them to have a common divisor in x, of a higher degree than the first. The degree of this divisor will depend upon the number of *multiple* values of x, which correspond to the same value of y. It will be of the second degree if there be two values of x to one of y; of the third, if three, &c. When therefore the divisor in the first degree becomes indeterminate, by the substitution of a value of y, deduced from the final equation, we make the substitution in the next superior divisor. If this divisor be of the second degree in x, and admit of solution, there will be *two* values of x corresponding to one value of y. If this equation also be indeterminate, we proceed to the next superior divisor, and, in general, to that divisor which does not reduce to an indeterminate form.

31. If we knew *à priori,* from the composition of the given equations, that they contained multiple values of x, for the same value of y, we might at once substitute the value of y in the divisor of the degree corresponding to the number of multiple roots; since its substitution in a divisor of an inferior degree would lead to an indeterminate result.

32. If *all* the values of y gave multiple values of x, the operation for obtaining the greatest common divisor would

necessarily stop at a divisor of a degree, corresponding to the number of these multiple roots; as is shown by the following example.

EXAMPLE IX.

$$x^4 + 2yx^3 + (2y^2 + 1)\,x^2 + (y^3 + 9y^2 + y - 81)\,x + y^2 = o$$

$$x^3 + 2yx^2 + 2y^2x + y^3 + 9y^2 - 81 = o.$$

First Division.

$$\begin{array}{c|l}
x^4+2yx^3+(2y^2+1)x^2+(y^3+9y^2+y-81)x+y^2 & x^3+2yx^2+2y^2x+y^3+9y^2-81 \\
\underline{x^4+2yx^3+2y^2x^2+xy^3+9y^2x-81x\qquad\qquad} & \overline{x\qquad\qquad\qquad\qquad\qquad} \\
x^2+yx+y^2 &
\end{array}$$

Second Division.

$$\begin{array}{l|l}
x^3 + 2yx^2 + 2y^2x + y^3 + 9y^2 - 81 & \underline{x^2 + yx + y^2} \\
\underline{x^3 + \ \ yx^2 + \ \ y^2x\qquad\qquad\qquad} & x + y \\
\qquad yx^2 + y^2x + y^3 & \\
\qquad \underline{yx^2 + y^2x + y^3\qquad} & \\
\qquad\qquad\qquad \underline{9y^2 - 81.} &
\end{array}$$

In this example the operation stops at a divisor of the second degree in x, the final equation being

$$9y^2 - 81 = o,$$

so that for each of the values of $y = \pm\, 3$, deduced from this equation, there are *two* values of x.

33. *The degree of the final cannot exceed the product of the numbers which represent the degrees of the given equations.*

M. Poisson demonstrates this principle in the following manner:

Let

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + u = o$$

$$x^n + P'x^{n-1} + Q'x^{n-2} + \ldots + T'x + u' = o,$$

be the two given equations; the coefficients P, Q, P′, Q′, &c. being functions of y of the most general form (Art. 1), as follows:

$$P = a + by,\ P' = a' + b'y,\ Q = c + dy + ey^2, \text{ \&c. \&c.}$$

If we substitute for P, P′, &c., their values, the above equations become

$$x^m + (a + by)\, x^{m-1} + (c + dy + ey^2)\, x^{m-2} \ldots + y^m = o$$

$$x^n + (a' + b'y)\, x^{n-1} + (c' + d'y + e'y^2)\, x^{n-2} \ldots + y^n = o.$$

But the degree of the final equation will not be diminished, if we reduce the coefficients of these last equations to the term which contains the highest power of y, since the degree of the two equations will not be changed by this operation. We shall then have

$$x^m + byx^{m-1} + ey^2x^{m-2} \ldots + y^m = o$$

$$x^n + b'yx^{n-1} + e'y^2x^{n-2} \ldots + y^n = o;$$

which may be placed under the form

$$\left(\frac{x}{y}\right)^m + b\left(\frac{x}{y}\right)^{m-1} + e\left(\frac{x}{y}\right)^{m-2} \ldots + 1 = o$$

$$\left(\frac{x}{y}\right)^n + b'\left(\frac{x}{y}\right)^{n-1} + e'\left(\frac{x}{y}\right)^{n-2} \ldots + 1 = o.$$

If we regard $\left(\frac{x}{y}\right)$ as the unknown quantity in these equations, and represent by α, β, γ, &c. the roots of $\left(\frac{x}{y}\right)$ in the first, and by α', β', γ', &c., those in the second equation, we shall have

$$\left(\frac{x}{y} - \alpha\right) \left(\frac{x}{y} - \beta\right) \left(\frac{x}{y} - \gamma\right) \text{ \&c.} = o$$

$$\left(\frac{x}{y} - \alpha'\right) \left(\frac{x}{y} - \beta'\right) \left(\frac{x}{y} - \gamma'\right) \text{ \&c.} = o;$$

or

$$(x - \alpha y)\ (x - \beta y)\ (x - \gamma y) \text{ \&c.} = o \ \ldots\ (1)$$

$$(x - \alpha' y)\ (x - \beta' y)\ (x - \gamma y) \text{ \&c.} = o \ \ldots\ (2).$$

If now we substitute in equation (1) each of the roots of x deduced from equation (2), viz:

$$x=\alpha' y \qquad x=\beta' y \qquad x=\gamma' y, \text{ \&c.},$$

we shall have n equation of the following form,

$$y^m(\alpha'-\alpha')\ (\alpha'-\beta)\ (\alpha'-\gamma) \text{ \&c.} = o$$

$$y^m(\beta'-\alpha)\ (\beta'-\beta)\ (\beta'-\gamma) \text{ \&c.} = o$$

$$y^m(\gamma'-\alpha)\ (\gamma'-\beta)\ (\gamma'-\gamma) \text{ \&c.} = o, \text{ \&c. \&c.}$$

each being of the mth degree, and giving m values of y. The whole number of values of y will therefore be $m \times n$, which will represent the degree of the final equation. The degree of the final equation cannot therefore exceed this number.

EXAMPLES.

$$1. \begin{cases} yx-y^2-y-1=o \\ yx-y^2-1=o. \end{cases}$$

Final equation, $-y=o$. Common divisor, $yx-y^2-1$. Equations *contradictory*. (See Art. 29.)

$$2. \begin{cases} x^2+(8y-13)\,x+y^2-7y+12=o \\ x^2-(4y+\ \ 1)\,x+y^2+5y=o. \end{cases}$$

Common divisor, $=(y-1)\ (12x-12)$. Final equation, $(y-1)\ (y^2+y)=o$.

Solutions.

$$y=1,\ y=1,\ y=o,\ y=-1$$

$$x=2,\ x=3,\ x=1,\ x=1.$$

3. $$\begin{cases} x^3-3yx^2+3x^2+3xy^2-6yx-x-y^3+3y^2+y-3=o \\ x^3+3yx^2-3x^2+3xy^2-6yx-x+y^3-3y^2-y+3=o. \end{cases}$$

The remainder of the second degree in x is divisible by $(y-1)$, which we suppress; after this suppression, the remainder of the first degree is divisible by y^2-2y. Final equation then becomes $y^2-2y-3=o$, and the common divisor x. (See Article 14.)

Solutions.

$$y=1,\ y=1,\ y=1,\ y=o,\ y=o,\ y=-1,\ y=2,\ y=2,\ y=3$$

$$x=o,\ x=2,\ x=-2,\ x=1,\ x=-1,\ x=o,\ x=1,\ x=-1,\ x=o.$$

4. $$\begin{cases} 3x^2-5yx^2-(3y^2-30y)\,x+30y^2=o \\ 6x^2-10y^2+11xy=o. \end{cases}$$

Final equation in x, $113x^4-1310x^3+1800x^2=o$.

Solutions.

$$x=o,\ x=o,\ x=10,\ x=\frac{180}{113}$$

$$y=o,\ y=o,\ y=15,\ y=-\frac{72}{113}.$$

$$5. \begin{cases} x^2 + y^2 - 5 = o \\ x^2 + \frac{5}{2}xy + y^2 = o. \end{cases}$$

Solutions.

$$y = 2,\ y = -2,\ y = 1,\ y = -1$$

$$x = -1,\ x = 1,\ x = -2,\ x = 2.$$

$$6. \begin{cases} x^2 + 2xy + y^2 - 1 = o \\ x^2 - \quad y^2 - 6y - 9 = o. \end{cases}$$

Solutions.

$$y = -1, \qquad y = -2$$

$$x = \quad 2, \qquad x = \quad 1.$$

$$7. \begin{cases} x^2 - 2yx + y^2 - 1 = o \\ x^2 + 2\,(y - 5)\,x + y^2 - 10y + 21 = o. \end{cases}$$

These equations can be placed under the following form :

$$\{x - (y + 1)\}\ \{x - (y - 1)\} = o$$

$$\{x - (3 - y)\}\ \{x - (7 - y)\} = o.$$

Solutions.

$$y = 1,\ y = 3,\ y = 2,\ y = 4$$

$$x = 2,\ x = 4,\ x = 1,\ x = 3.$$

$$8.\ \begin{cases} x^2 - 2yx - 4x + y^2 + 4y + 3 = o \\ x^2 - 3yx - 5y + 2y^2 - 11y - 6 = o. \end{cases}$$

These equations may be written thus :

$$\{x - (y + 1)\} \quad \{x - (3 + y)\} = o$$

$$\{x - (2y - 1)\} \ \{x - (6 + y)\} = o.$$

Putting these factors two and two equal to zero, we have

$$x = 3,\ x = 7, \qquad 1 = 6,\ 3 = 6,$$

$$y = 2,\ y = 4.$$

The two last results are absurd. Final equation, $y^2 - 6y + 8 = o$.

$$9.\ \begin{cases} x^2 - 4yx + 4y^2 - 1 = o \\ x^2 - (4y + 5)\,x + 2\,(2y^2 + 5y + 3) = o. \end{cases}$$

Which may be placed under the following form :

$$\{x - (2y + 1)\} \ \{x - (2y - 1)\} = o$$

$$\{x - (2y + 3)\} \ \{x - (2y + 2)\} = o\,;$$

which furnish the following absurd results :

$$1 = 3, \quad 1 = 2, \quad -1 = 3, \quad -1 = 2.$$

If we had applied the ordinary rule for determining the final equation, we should have found a numerical remainder (see Art. 25.)

$$10. \quad \begin{cases} (x - y)\,(x + y - 1)\,(x + y + 1) = o \\ (x - y)^{\cdot}\,(x - y - 3)\,(x + y + 3). \end{cases}$$

Suppressing the common factor $(x - y)$, Art. 14, we have the following systems of equations:

(1) $x + y - 1 = o$ and $x - y - 3 = o$

(2) $x + y + 1 = o$ and $x - y - 3 = o$

(3) $x + y - 1 = o$ and $x + y + 3 = o$

(4) $x + y + 1 = o$ and $x + y + 3 = o$.

Equations (3) and (4) are contradictory.

$$11. \quad \begin{cases} x^2 - 2yx + 8 = o \\ x^2 - 2y^2 + 14 = o. \end{cases}$$

Final equation, $y^4 - 8y^2 - 9 = o$. Common divisor, $yx - y^2 + 3$.

Solutions.

$$y = 3,\ y = -3,\ y = +\sqrt{-1},\ y = -\sqrt{-1}$$

$$x = 2,\ x = -2,\ x = 4\sqrt{-1},\ x = -4\sqrt{-1},$$

12. $\begin{cases} (y-1)\,x^3 + y\,(y+1)\,x^2 + (3y^2 + y - 2)\,x + 2y = o \\ (y-1)\,x^2 + y\,(y+1)\,x + 3y^2 - 1 = o. \end{cases}$

Final equations, $y^2 - 1 = o$. Common divisor, $(y-1)$ $x + 2y$.

The value of $y = 1$ must be rejected (Art. 29,) since it reduces the common divisor to 2.

Solutions.

$$y = -1, \qquad x = -1,$$

ANALYSIS OF CURVES.

1. We have seen in *Analytical Geometry*, that every equation between two indeterminates, may be conceived to express the relation between the abscissas and ordinates of the curve, which this equation represents. By giving particular values to either of the variables, the corresponding values of the other may be deduced, and all the points of the curve determined. Following the course therein defined, we may ascertain whether or not this curve is symmetrical with respect to either or both the co-ordinate axes; we may also define its limits when any exist, by determining the points at which the tangents are parallel to the axes, or by ascertaining the existence and position of its asymptotes.

2. Beyond this, however, the powers of Analytical Geometry end, and we are compelled to resort to those means which the discovery of the science of the *Differential Calculus* has placed at our command. By this we may not only verify the *results* of the geometrical analysis, but we may trace with the most exact certainty, the course of any curve however irregular, and define its properties however peculiar. The sole difficulty consists in solving the algebraic equation which defines the curve. If this difficulty be removed, we may readily trace its course. For, suppose that the equation of the curve has been solved, and that X, X',

X$''$, &c. represent the roots of y, these roots being functions of x; the question is at once reduced to an examination of the particular curves, which are expressed by the separate equations

$$y = \mathrm{X}, \; y = \mathrm{X}', \; y = \mathrm{X}'', \; \&c.$$

This examination will be effected by giving to x every possible value, as well negative as positive, which the functions X, X$'$, X$''$, &c. admit of, without becoming imaginary; and the curves which result will be the different *branches* of the curve represented by the given equation. The extent of each of these branches will depend upon the different solutions which correspond to its particular equation. If any of the equations

$$y = \mathrm{X}, \; y = \mathrm{X}', \; y = \mathrm{X}'',$$

exist for infinite values of x, it follows that these branches extend indefinitely in the direction of these values. Let us apply these principles to the analysis of the

Lemniscate Curve.

3. Take the equation

$$y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = o.$$

This being a quadratic equation, its solution is effected by

the ordinary rules for such equations, and we find the values of y to be

$$y = \pm\sqrt{48a^2 \pm \sqrt{2304a^4 - 100a^2x^2 + x^4}}$$

or putting

$$2304a^4 - 100a^2x^2 + x^4 = \mathrm{N},$$

the four values of y become

$$y = \sqrt{48a^2 + \sqrt{\mathrm{N}}} \;..\; (1) \qquad y = \sqrt{48a^2 - \sqrt{\mathrm{N}}} \;..\; (2)$$

$$y = -\sqrt{48a^2 + \sqrt{\mathrm{N}}} \;.\; (3) \qquad y = -\sqrt{48a^2 - \sqrt{\mathrm{N}}} \;.\; (4)$$

It is required now to ascertain each of the curves which these equations represent.

We see in the first place, that the values (3) and (4) only differ from those of (1) and (2) in the *sign*, and consequently must represent similar branches whose position with respect to the axis of x alone differs. Further, as the quantity N contains *even* powers of x only, its value will not be changed by substituting a negative for a positive value of x. The parts of the curve which lie on the right of the axis of y, are therefore similar to those which lie on the left of this

axis. Hence the curve is divided by the co-ordinate axes into four equal and symmetrical parts.

4. Let us now examine more particularly the values (1) and (2).

They can only be real so long as the quantity N is positive; the limit to the real values of y will then be found by making

$$N = x^4 - 100a^2x^2 + 2304a^4 = o.$$

But this equation can be decomposed into the factors

$$x - 6a,\ x + 6a,\ x - 8a,\ x + 8a,$$

and the values of y for equation (1) will be

$$y = \sqrt{48a^2 + \sqrt{(x - 6a)(x + 6a)(x - 8a)\ (x + 8a)}}$$

For any values of x greater than $6a$, but less than $8a$, the values of y will be imaginary, since the factor $(x - 8a)$ under this supposition is negative. No part of the curve then is embraced within the limits

$$x = 6a,\ x = 8a;$$

but for values of x greater than $8a$, the factor $(x - 8a)$ becomes positive, and the values of y always real.

The values of y which correspond to the three values of x,

$$x = o,\ x = 6a,\ x = 8a,$$

will be found from equation (1) to be

$$y = \sqrt{96a^2},\quad y = \sqrt{48a^2},\quad y = \sqrt{48a^2}.$$

Equation (1) gives then, 1st, a part DF (see Figure 1) which extends from the point D, taken on the axis AC, to the point F, whose abscissa AE $= 6a$; 2ndly, a part HX, which beginning at the point H, whose abscissa AG $= 8a$, extends indefinitely in the angle BAC.

5. Equation (2), which, when the factors of N are introduced, becomes

$$y = \sqrt{48a^2 - \sqrt{(x - 6a)(x + 6a)(x - 8a)(x + 8a)}},$$

will in like manner give imaginary results *between* the limits

$$x = 6a,\quad x = 8a;$$

but for the values

$$x = o,\ x = 6a.\ x = 8a,$$

we get

$$y = o,\ y = \sqrt{48a^2},\ y = \sqrt{48a^2},$$

which show, 1st, that equation (2) gives a part AF, which unites with the part DF given by equation (1) at the point F, for which the two ordinates are equal; 2ndly, beginning at the point H, equation (2) gives a part HK, in which y decreases until $\sqrt{N} = 48a^2$, when it becomes zero, and corresponds to the point I. For N greater than $48a^2$ the quantity under the radical becomes negative, and y imaginary. The Branch of the Curve corresponding to equation (2) does not therefore extend beyond the point I. The abscissa of this point is evidently determined by making $y = o$ in equation (2). We find

$$x = \pm\, o,\quad x = \pm\, 10a.$$

The two first values correspond to the point A, the others to the points I and I'.

6. We might continue this discussion, which is in every respect analogous to the general discussion of an equation of the second degree in analytical geometry, and ascertain whether this curve has asymptotes; but as the differential calculus abridges this investigation, we will at once apply it to this purpose, and then proceed to the determination of the singular points of the curve.

7. An examination of the four values of y, Arts. 3 and 4,

Fig. 1.

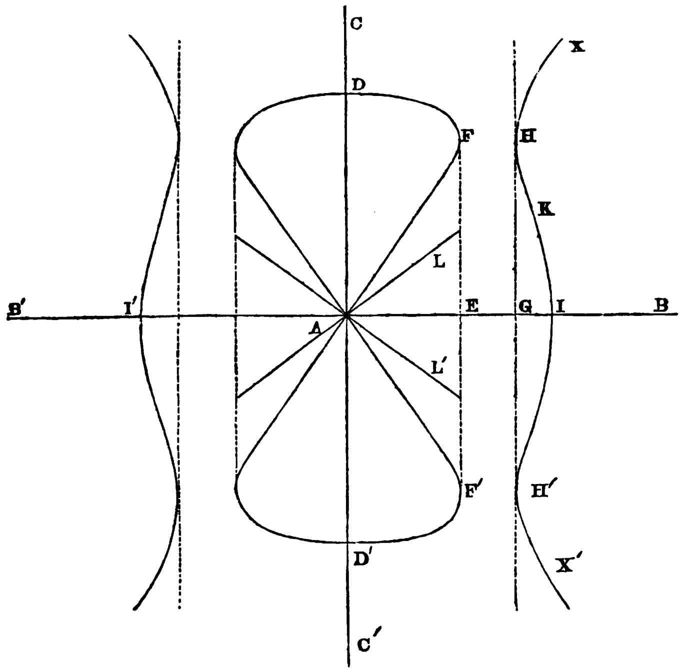
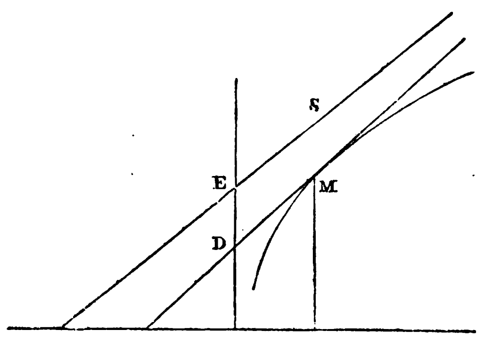


Fig. 2.



has already shown that the curve we are discussing has in each angle of the co-ordinate axes an indefinite branch. Let us see whether these branches have asymptotes.

We know that if any curve MX (Fig. 2) have an asymptote RS, the tangent MT approaches more and more a coincidence with the asymptote as the point of tangency is removed from the origin. Under this supposition, the points T and D in which the tangent intersects the axes, will continually approach the points R and E, in which the asymptotes intersect the axes; so that AR and AE are limits to the values of AT and AD. Hence, to ascertain whether a curve has asymptotes, it is necessary to determine whether the expressions AT and AD, which represent the distances from the origin to the points in which the tangent cuts the co-ordinate axes, have limits for infinite values of x and y. If they have, these limits being constructed will give the points D and E, through which, if the line RS be drawn, it will be the asymptote sought.

8. The expressions for AT and AD may be deduced at once from the equation of the tangent line. The equation of the tangent line is

$$y - y' = \frac{dy}{dx}(x - x'),$$

$\frac{dy}{dx}$ being the tangent of the angle which it makes with the axis of x. We may now obtain the distances AT and AD, by making y' and x' separately equal to zero. By the first supposition we have

$$x' = AT = x - y\,\frac{dx}{dy},$$

in which the quantity $y\frac{dx}{dy}$, which is the expression for the subtangent PT, is taken *negatively*, since it is counted in an opposite direction from the abscissa x'. Making now $x' = o$, we have

$$y' = AD = y - x\,\frac{dy}{dx}.$$

9. To ascertain whether the given curve has asymptotes, we must substitute the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$, deduced from the equation of the curve, in the expressions for AT and AD, and see what these expressions become when x and y are infinite. We find the first differential co-efficient of the given equation after dividing by 4, to be

$$\frac{dx}{dy} = \frac{y^3 - 48a^2y}{x^3 - 50a^2x}.$$

Multiplying this value by y, and subtracting the product from x, we have after reducing,

$$AT = x - y\,\frac{dx}{dy} = \frac{x^4 - 50a^2x^2 - y^4 + 48a^2y^2}{x^3 - 50a^2x}.$$

By a simple transposition of the terms of the fraction,

which forms the value of $\frac{dx}{dy}$, we deduce that of $\frac{dy}{dx}$, and we have

$$AD = y - x\,\frac{dy}{dx} = \frac{y^4 - 48a^2y^2 - x^4 + 50a^2x^2.}{y^3 - 48a^2y}$$

Putting in these expressions the value of y^4, they become

$$AT = \frac{50a^2x^2 - 48a^2y^2}{x^3 - 50a^2x},$$

$$AD = \frac{48a^2y^2 - 50a^2x^2}{y^3 - 48a^2y}.$$

These values of AT and AD, continually diminish as x and y increase, and when x and y equal $\pm$ infinity, they become zero. We conclude, then, that the curve has two asymptotes, which pass through the origin of co-ordinates. Their angle is determined by seeing what $\frac{dy}{dx}$ becomes when x and y equal $\pm$ infinity. We have

$$\frac{dy}{dx} = \frac{x^3 - 50a^2x}{y^3 - 48a^2y}$$

when x and y are infinite, the first powers of y and x may be neglected, and we have

$$\frac{dy}{dx} = \pm 1,$$

which shows that one of the asymptotes makes an angle with the axis of x of 45° ; the other an angle of 45° + 90° = 135°.

10. Let us now examine the singular points of the curve. We find the first differential co-efficient to be

$$\frac{dy}{dx} = \frac{x^3 - 50a^2x}{y^3 - 48a^2y}.$$

To determine the points at which the tangent is parallel to the axis of x, make $\frac{dy}{dx} \pm o$. We find

$$x^3 - 50a^2x = o,$$

which gives

$$x = o \quad x = + \sqrt{50a^2} \quad x = - \sqrt{50a^2}$$

The value of $x = o$, when substituted in the given equation, gives

$$y = \pm o \quad y = \pm \sqrt{96a^2}.$$

But when $x = o$ and $x = o$, we have

$$\frac{dy}{dx} = \frac{x^3 - 50a^2x}{y^3 - 48a^2y} = \frac{o}{o}$$

which indicates a *multiple* point at the origin of co-ordinates, (see Boucharlat's Differential Calculus, Art. 138.)

To determine the value of $\frac{dy}{dx}$, we must pass to the second differential equation, which becomes, when $x = o$ and $y = o$,

$$-48a^2dy^2 + 50a^2dx^2 = o.$$

Hence

$$\frac{dy}{dx} = \pm\sqrt{\frac{50}{48}}.$$

It follows from these values, that at the point A, the curve is touched by two straight lines, which make angles with the axis of x, the tangents of which are

$$+\sqrt{\frac{50}{48}}, \text{ and } -\sqrt{\frac{50}{48}};$$

the point A is therefore a multiple point.

11. The values of $y = \pm\sqrt{96a^2}$ correspond to the points D and D', at which the tangent is parallel to the axis of x.

12. If we deduce the 3d differential equation, we find by making $x = o, y = o$,

$$-48a^4\,dy\,d^2y - 96a^2dy\,d^2y = o,$$

or

$$+ 144a^2 \, dy \, d^2y = o;$$

from which we conclude that

$$\frac{d^2y}{dx^2} = o.$$

The second differential co-efficient being zero, let us find the third differential co-efficient from the fourth differential equation. This equation, when we make $x = o$, $y = o$, and $\frac{d^2y}{dx^2} = o$, reduces to

$$- 4.48a^2 \, dy \, d^3y + 6dy^4 - 6dx^4 = o;$$

from which we deduce

$$- 32a^2 \frac{dy}{dx} \frac{d^3y}{dx^3} + \frac{dy^4}{dx^4} - 1 = o.$$

Hence

$$\frac{d^3y}{dx^3} = \frac{\left(\frac{50}{48}\right)^2 - 1}{\pm 32a^2 \sqrt{\frac{50}{48}}}.$$

when $\frac{dy}{dx}$ is replaced by its value $\pm\sqrt{\frac{50}{48}}$.

But we have found (Boucharlat, Art. 123), for the distance between the tangent and the curve,

$$\delta = \frac{d^2y\,h^2}{dx^2 1.2} + \frac{d^3y\,h^3}{dx^3 1.2.3}\ \&c.$$

When we make in this expression

$$\frac{d^2y}{dx^2} = o,$$

and

$$\frac{d^3y}{dx^3} = \frac{\left(\frac{50}{48}\right)^2 - 1}{\pm\, 32a^2\sqrt{\frac{50}{48}}}$$

it becomes

$$\delta = \pm\frac{\left(\frac{50}{48}\right)^2 - 1}{32a^2\sqrt{\frac{50}{48}}} \times \frac{h^3}{1.2.3} + \&c.$$

which shows that the branch of the curve touched by the straight line AL, which corresponds to the positive value of

$\frac{dy}{dx}$, is above the straight line on the side of the positive abscissas; and below it, on the side of the negative abscissas. The reverse takes with respect to the tangent AL′. Hence each branch of the curve undergoes an *inflexion* at the point A.

13. If in the 2nd differential equation

$$3y^2dy^2+y^3d^2y-48a^2dy^2-48a^2yd^2y+50a^2dx^2+50a^2xd^2x-3x^2dx^2-x^3d^2x=o,$$

we make $x=o$, and $y=\pm\sqrt{96a^2}$ and $\frac{dy}{dx}=o$, we have

$$\frac{d^2y}{dx^2}=-\frac{50a^2}{(\pm\sqrt{96a^2})^2-48a^2\times\pm\sqrt{96a^2}}.$$

which gives a negative value for $\frac{d^2y}{dx^2}$ for the value of y, corresponding to the point D, and a positive value for D′, which shows that at D, the ordinate is a *maximum*, while at D′ it is a *minimum*. The ordinate at D′ is regarded as a minimum, because every increment to a negative ordinate is equivalent to a decrement with respect to positive ordinates.

14. To find the points at which the tangents are perpendicular to the axes of x, we must put $\frac{dy}{dx}=\theta$, or what is equivalent to it, place the denominator of its value equal to zero. This gives